

**TRANSFER TO CHARACTERISTIC ZERO: APPENDIX TO
“FUNDAMENTAL LEMMA OF JACQUET-RALLIS IN POSITIVE
CHARACTERISTICS” BY ZHIWEI YUN**

JULIA GORDON

The purpose of this Appendix is to point out that the work of R. Cluckers, T. Hales and F. Loeser [4] implies that Transfer principle of Cluckers and Loeser [3] applies to the version of the Fundamental Lemma proved in [8]. Thus, the Conjectures 1.1.1 and 1.1.2 of [8] are true when F is a local field of characteristic zero, with sufficiently large residue characteristic.

We need to emphasize that even though these Conjectures in the equal characteristic case are proved in [8] for the fields F of characteristic larger than n , the transfer principle leads to a slightly weaker result for the fields of characteristic zero. Namely, there exists (an algorithmically computable) constant M such that the Conjectures hold for the characteristic zero local fields F of residue characteristic larger than M . However, we hope that such a result is sufficient for some applications.

Since this appendix is of an expository nature, the references often point not to the original sources, but to more expository articles. All the references to specific sections, conjectures, and definitions that do not mention a source are to Z. Yun’s article for which this appendix is written.

Acknowledgement. This appendix emerged as a result of the AIM workshop on Relative Trace Formula and Periods of Automorphic Forms in August 2009. It is a pleasure to thank the organizers and participants of this workshop. I would like to emphasize that all the original ideas used and described here appear in the works of R. Cluckers, T.C. Hales, and F. Loeser. I am grateful to R. Cluckers for a careful reading.

1. DENEFF-PAS LANGUAGE

The idea behind the approach to transfer described here is to express everything involved in the statement of the Fundamental Lemma by means of formulas in a certain first-order language of logic (called the Denef-Pas language) \mathcal{L}_{DP} (see, e.g., [4, Section 1.6] for the detailed definition), and then work with these formulas directly instead of the sets and functions described by them. Denef-Pas language is designed for valued fields. It is a *three-sorted language*, meaning that it has three sorts of variables. Variables of the first sort run over the valued field, variables of the second sort run over the value group (for simplicity, we shall assume that the value group is \mathbb{Z}), and variables of the third sort run over the residue field.

Let us describe the set of symbols that, along with parentheses, the binary relation symbol ‘=’ in every sort, the standard logical symbols for conjunction, disjunction, and negation, and the quantifiers, are used to build formulas in Denef-Pas language.

- In the valued field sort: there are constant symbols ‘0’ and ‘1’, and the symbols ‘+’ and ‘×’ for the binary operations of addition and multiplication. Additionally, there are symbols for two functions from the valued field sort: ‘ord(·)’ to denote a function from the valued field sort to the \mathbb{Z} -sort, and ‘ $\overline{\text{ac}}(\cdot)$ ’ to denote a function from the valued field sort to the residue field sort. These functions are called the *valuation map*, and the *angular component map*, respectively.
- In the residue field sort: there are constant symbols ‘0’ and ‘1’, and the binary operations symbols ‘+’ and ‘×’ (thus, restricted to the residue field sort, this is the language of rings).
- In the \mathbb{Z} -sort, there are ‘0’ and ‘1’, and the operation ‘+’; additionally, for each $d = 2, 3, 4, \dots$, there is a symbol ‘ \equiv_d ’ to denote the binary relation $x \equiv y \pmod{d}$. Finally, there is a binary relation symbol ‘ \geq ’. (This is Presburger language for the integers).

Given a discretely valued field K with a uniformizer of the valuation ϖ , the functions ‘ord(·)’ and ‘ $\overline{\text{ac}}(\cdot)$ ’ are interpreted as follows. The function $\text{ord}(x)$ stands for the valuation of x . It is in order to provide the interpretation for the symbol ‘ $\overline{\text{ac}}(x)$ ’ that a choice of the uniformizing parameter ϖ (so that $\text{ord}(\varpi) = 1$) is needed. If $x \in \mathcal{O}_K^*$ is a unit, there is a natural definition of $\overline{\text{ac}}(x)$ – it is the reduction of x modulo the ideal (ϖ) . For $x \neq 0$ in K , $\overline{\text{ac}}(x)$ is defined by $\overline{\text{ac}}(x) = \overline{\text{ac}}(\varpi^{-\text{ord}(x)}x)$, and by definition, $\overline{\text{ac}}(0) = 0$.

A formula φ in \mathcal{L}_{DP} can be interpreted in any discretely valued field, once a uniformizer of the valuation is chosen, in the sense that given a valued field K with a uniformizer ϖ and the residue field k_K , one can allow the free variables of φ to range over K , k_K , and \mathbb{Z} , respectively, according to their sort (naturally, the variables bound by a quantifier then also range over K , k_K , and \mathbb{Z} , respectively). Thus, any discretely valued field is a structure for Denef-Pas language.

2. CONSTRUCTIBLE MOTIVIC FUNCTIONS

In the foundational papers [2], [3], R. Cluckers and F. Loeser developed the theory of motivic integration for functions defined by means of formulas in Denef-Pas language, and proved a very general Transfer Principle. We refer to [4] and [1] for the introduction to this subject and all definitions. Note that the article [4] is self-contained and essentially covers everything in this appendix.

Here we need to use the terms “definable subassignment” and “constructible motivic function”. Let $h[m, n, r]$ be the functor from the category of fields to the category of sets defined by

$$h[m, n, r](K) = K((t))^m \times K^n \times \mathbb{Z}^r.$$

The term *subassignment* was first introduced in [5]. Given a functor F from some category \mathcal{C} to Sets, a subassignment X of F is a collection of subsets $X(A) \subset F(A)$ for every object A of \mathcal{C} . A definable set is a set that can be described by a formula in Denef-Pas language, and a subassignment X of the functor $h[m, n, r]$ is called definable if there exists a formula φ in Denef-Pas language with m free variables of the valued field sort, n free variables of the residue field sort, and r free variables of the value group sort, such that for every field K , the set $X(K)$ is exactly the set of points in $K((t))^m \times K^n \times \mathbb{Z}^r$ where φ takes the value ‘true’. Note that there are slightly different variants of Denef-Pas language, depending

on the sets of coefficients for a formula φ allowed in every sort (the smallest set of coefficients is \mathbb{Z} in every sort; however, one can add constant symbols that can later be used as coefficients – one such variant will be discussed below). We emphasize, however, that regardless of the variant, the coefficients come from a fixed set, and are independent of K . Definable subassignments form a Boolean algebra in a natural way, and this algebra is the replacement, in the theory of motivic integration, for the Boolean algebra of measurable sets in the traditional measure theory.

For a definable subassignment X , the ring of the so-called *constructible motivic functions* on X , denoted by $\mathcal{C}(X)$, is defined in [2]. The elements of $\mathcal{C}(X)$ are, essentially, formal constructions defined using the language \mathcal{L}_{DP} . The main feature of constructible motivic functions is specialization to functions on discretely valued fields. Namely, let $f \in \mathcal{C}(X)$. Let F be a non-Archimedean local field (either of characteristic zero or of positive characteristic). Let ϖ be the uniformizer of the valuation on F . Given these data, one gets a specialization X_F of the subassignment X to F , which is a definable subset of $F^m \times k_F^n \times \mathbb{Z}^r$ for some m, n, r , and the constructible motivic function f specializes to a \mathbb{Q} -valued function f_F on X_F , for all fields F of residue characteristic bigger than a constant that depends only on the \mathcal{L}_{DP} -formulas defining f and X . As explained in [4, Section 2.9], one can tensor the ring $\mathcal{C}(X)$ with \mathbb{C} , and then the specializations f_F of elements of $\mathcal{C}(X) \otimes \mathbb{C}$ form a \mathbb{C} -algebra of functions on X_F .

3. INTEGRATION AND TRANSFER PRINCIPLE

In [2], Cluckers and Loeser defined a class $\text{IC}(X)$ of *integrable* constructible motivic functions, closed under integration with respect to parameters (where integration is with respect to the *motivic measure*). Given a local field F with a choice of the uniformizer, these functions specialize to integrable (in the classical sense) functions on X_F , and motivic integration specializes to the classical integration with respect to an appropriate Haar measure, when the residue characteristic of F is sufficiently large.

From now on, we will use the variant of the theory of motivic integration with coefficients in the ring of integers of a given global field. Let Ω be a global field with the ring of integers \mathcal{O} . Following [1], we denote by $\mathcal{A}_{\mathcal{O}}$ the collection of all p -adic completions of all finite extensions of Ω , and by $\mathcal{B}_{\mathcal{O}}$ the set of all local fields of positive characteristic that are \mathcal{O} -algebras. Let $\mathcal{A}_{\mathcal{O},M}$ (resp., $\mathcal{B}_{\mathcal{O},M}$) be the set of all local fields F in $\mathcal{A}_{\mathcal{O}}$ (resp., $\mathcal{B}_{\mathcal{O}}$) such that the residue field k_F has characteristic larger than M . Let $\mathcal{L}_{\mathcal{O}}$ be the variant of Denef-Pas language with coefficients in $\mathcal{O}[[t]]$ (see [1, Section 6.7] for the precise definition). This means, roughly, that a constant symbol for every element of $\mathcal{O}[[t]]$ is added to the valued field sort, so that a formula in $\mathcal{L}_{\mathcal{O}}$ is allowed to have coefficients in $\mathcal{O}[[t]]$ in the valued field sort, coefficients in Ω in the residue field sort, and coefficients in \mathbb{Z} in the value group sort.

Then the Transfer Principle can be stated as follows.

Theorem 1. (*Abstract Transfer Principle, cf. [4, Theorem 2.7.2]*). *Let X be a definable subassignment, and let φ be a constructible (with respect to the language $\mathcal{L}_{\mathcal{O}}$) motivic function on X . Then there exists $M > 0$ such that for every $K_1, K_2 \in$*

$\mathcal{A}_{\mathcal{O},M} \cup \mathcal{B}_{\mathcal{O},M}$ with $k_{K_1} \simeq k_{K_2}$,

$$\varphi_{K_1} = 0 \quad \text{if and only if} \quad \varphi_{K_2} = 0.$$

Remark 2. In fact, the transfer principle is proved in [3] for an even richer class of functions, called constructible motivic exponential functions, that contains additive characters of the field along with the constructible motivic functions. However, we do not discuss it here since the characters are not needed in the present setting.

The goal of this appendix is to check that the Conjectures proved in this article can be expressed as equalities between specializations of constructible motivic functions. We emphasize that all the required work is actually done in [4], here we just check that it indeed applies in the present situation.

4. DEFINABILITY OF ALL THE INGREDIENTS

Here we go through Section 2.1 and check that every object appearing in it is definable.

4.1. The degree two algebra E/F . Following [4, Section 4], we fix, once and for all, a \mathbb{Q} -vector space V of dimension n , and fix a basis e_0, \dots, e_n of V over \mathbb{Q} .

As in [4, Section 3.2], we introduce a parameter (which we denote by ϵ) that will appear in all the formulas that involve an unramified quadratic extension of the base field. We think of ϵ as a non-square unit, and denote by Λ be the subassignment of $h[1, 0, 0]$ defined by the formula ‘ $\text{ord}(\epsilon) = 0 \wedge \nexists x : x^2 = \epsilon$ ’. From now on, we only consider the relative situation: all the subassignments we consider will come with a fixed projection morphism to Λ (in short, we are considering the category of definable subassignments over Λ , see [2, Section 2.1]). That is, we replace all the constructions that depend on an unramified quadratic extension E/F (such as the unitary group), with the family of isomorphic objects parameterised by a non-square unit ϵ in F . Now, imagine that we fixed the basis $(1, \sqrt{\epsilon})$ for the quadratic extension E . Then E can be identified with F^2 via this basis; so from now on we shall think of the elements of E as pairs of variables that range over F . The nontrivial Galois automorphism σ of E over F now can be expressed as a 2×2 -matrix with entries in F , and can be used in the expressions in Denef-Pas language.

The nontrivial quadratic character $\eta_{E/F}$ can be expressed by a Denef-Pas formula ‘ $\eta_{E/F}(x) = 1 \Leftrightarrow \exists (a, b) \in F^2 : (a^2 + \epsilon b^2 = x)$ ’, or, simply, ‘ $\eta_{E/F}(x) = 1 \Leftrightarrow \text{ord}(x) \equiv 0 \pmod{2}$ ’.

In the case E/F split, we just treat elements of E as pairs of elements of F .

4.2. The groups and their Lie algebras. In Sections 2.1, 2.2, one starts out with free \mathcal{O}_F -modules W and V , and then proceeds to choose a basis vector e_0 with certain properties. We shall reverse the thinking here: we fix a basis e_0, e_1, \dots, e_n , and fix the dual basis e_0^*, \dots, e_n^* , such that $e_0^*(e_0) = 1$, and such that the Hermitian form (\cdot, \cdot) on V , with respect to this basis, corresponds to a matrix with entries in the set $\{0, \pm 1\}$, and $(e_0, e_0) = 1$, and $(e_0, e_j) = 0$ for $1 \leq j \leq n$. We let W be the span of the vectors e_1, \dots, e_{n-1} . With this choice of basis, we think of the elements of \mathfrak{gl}_n as n^2 -tuples of variables $A = (a_{ij})$. (Formally speaking, we identify \mathfrak{gl}_n with the definable subassignment $h[n^2, 0, 0]$.) All the split algebraic groups are, naturally, defined by polynomial equations in these variables, and thus can be

replaced with definable subassignments of $h[n^2, 0, 0]$. The embedding $\mathrm{GL}_{n-1} \hookrightarrow \mathrm{GL}_n$ where

$$A \mapsto \begin{pmatrix} A & \\ & 1 \end{pmatrix}$$

is, clearly, definable.

To find the definable subassignments that specialize to \mathfrak{s}_n , \mathfrak{u}_n , and U_n , we introduce the parameter ϵ as above in Section 4.1. Then \mathfrak{s}_n naturally becomes a definable subassignment of $h[2n^2, 0, 0] \times \Lambda \subset h[2n^2 + 1, 0, 0]$. Indeed, as discussed above, the Galois automorphism σ can be used in $\mathcal{L}_{\mathrm{DP}}$ -expressions when we think of the elements of E as pairs of F -variables: we replace each variable a_{ij} ranging over E with a pair of variables (x_{ij}, y_{ij}) ranging over F . The Hermitian form that is used to define the unitary group, given the choice of the basis, gives rise to polynomial equations in (x_{ij}, y_{ij}) that define the unitary group. Hence, \mathfrak{u}_n and U_n can also be replaced with definable subassignments of $h[2n^2 + 1, 0, 0]$.

4.3. The invariants. By definition, $a_i(A)$ are the coefficients of the characteristic polynomial of A , and in particular, they are polynomial expressions in the matrix entries of A , and therefore they are given by terms in $\mathcal{L}_{\mathrm{DP}}$, and the map $A \mapsto (a_i(A))_{0 \leq i \leq n-1}$ is definable (recall that a function is called definable if its graph is a definable set).

First, let us consider the case when E/F is a field extension.

The linear functional e_0^* on V (defined in Section 2.1), with our choice of the bases, is just the covector $(1, 0, \dots, 0)$. Then the invariants $b_i(A)$ of Section 2.2 are also given by terms in $\mathcal{L}_{\mathrm{DP}}$.

The vectors $A^i e_0$ are, clearly, just columns of polynomial expressions in the matrix entries (x_{ij}, y_{ij}) of A . The condition that a collection of vectors forms a basis of a given vector space is a predicate in $\mathcal{L}_{\mathrm{DP}}$. Hence, the set of semisimple elements in $\mathfrak{gl}_n(E)$ that are strongly regular with respect to $\mathrm{GL}_{n-1}(E)$ -action (in the sense of Definition 2.2.1) is a specialization (to F) of a definable subassignment of $h[2n^2 + 1, 0, 0]$.

We observe that $\Delta_{a,b} = \det(e_0^* A^{i+j} e_0)_{0 \leq i,j \leq n}$ (of Definition 2.2.3) is also a polynomial expression in (x_{ij}, y_{ij}) .

Recall the subassignment Λ from the previous subsection that specializes to the domain for a parameter ϵ defining the extension E . Since the image of a definable subassignment under a definable morphism is a definable subassignment, we have the definable subassignment \mathcal{P} over Λ , which we will denote by $\mathcal{P} \rightarrow \Lambda$, that corresponds to the set of pairs $(a, b) \in E^{2n}$ that are invariants of some strongly regular element of $\mathfrak{gl}_n(E)$. More precisely, \mathcal{P} is a subassignment of $\Lambda \times h[4n, 0, 0]$ that satisfies the condition that there exists $N > 0$, such that for every local field $F \in \mathcal{A}_{\mathcal{O}, N} \cup \mathcal{B}_{\mathcal{O}, N}$, for every $\epsilon \in \Lambda_F$, the fibre \mathcal{P}_ϵ of \mathcal{P} at ϵ specializes to the set of pairs (a, b) that are invariants of some $A \in \mathfrak{gl}_n(E)$, strongly regular with respect to $\mathrm{GL}_{n-1}(E)$ -action (in the sense of Definition 2.2.1), where E is the field extension corresponding to ϵ .

If E/F is split, the same argument works, except there is no need to consider the relative situation over Λ .

Since we have a symbol for the F -valuation in $\mathcal{L}_{\mathrm{DP}}$, the parameter $\nu(A)$ of Definition 2.2.2 is also an expression in $\mathcal{L}_{\mathrm{DP}}$.

4.4. The orbital integrals. Since the quadratic character $\eta_{E/F}$ only takes the values ± 1 , we can break the orbital integral $O_A^{\mathrm{GL}_{n-1}, \eta}(\mathbf{1}_{\mathfrak{s}_n(\mathcal{O}_F)})$ into the difference of two integrals:

$$\begin{aligned} O_A^{\mathrm{GL}_{n-1}, \eta}(\mathbf{1}_{\mathfrak{s}_n(\mathcal{O}_F)}) &= \int_{\mathrm{GL}_{n-1}(F) \cap \{g | \eta_{E/F}(\det g) = 1\}} \mathbf{1}_{\mathfrak{s}_n(\mathcal{O}_F)}(g^{-1}Ag) dg \\ &\quad - \int_{\mathrm{GL}_{n-1}(F) \cap \{g | \eta_{E/F}(\det g) = -1\}} \mathbf{1}_{\mathfrak{s}_n(\mathcal{O}_F)}(g^{-1}Ag) dg. \end{aligned}$$

By the remarks in Section 4.1 above, both domains of integration are definable sets. For each point A in the subassignment of strongly regular elements, $\mathbf{1}_{\mathfrak{s}_n(\mathcal{O}_F)}(g^{-1}Ag)$ is, by Section 4.2 above, a specialization of a constructible motivic function of g . We need to briefly discuss the normalization of the measures. The p -adic measure to which the motivic measure specializes is the so-called Serre-Oesterlé measure, defined in [7]. Serre-Oesterlé measure on a classical group G is the Haar measure such that the volume of the maximal compact subgroup is $q^{\dim G}$. Hence, the Haar measure dg differs from Serre-Oesterlé measure by a factor of $q^{-(n-1)^2}$, where q is the cardinality of the residue field, since, as in [6], the Haar measures here are chosen so that the standard maximal compact subgroups have volume 1. This factor is the specialization of the (constant) constructible motivic function $\mathbb{L}^{-(n-1)^2}$ (see e.g. [4, Section 2.3] for the discussion of the symbol \mathbb{L}). We conclude that $O_A^{\mathrm{GL}_{n-1}, \eta}(\mathbf{1}_{\mathfrak{s}_n(\mathcal{O}_F)})$ is a specialization of a constructible motivic function of A .

By a similar inspection, we see that the integral $O_{A'}^{U_{n-1}}(\mathbf{1}_{\mathfrak{u}_n(\mathcal{O}_F)})$ is a specialization of a constructible motivic function of A' , and thus so is the right-hand side of Conjecture 1.1.1 (1).

Finally, recall the subassignment \mathcal{P} from Section 4.3 that specializes to the set of invariants. Consider the subassignment \mathcal{X} of $\mathfrak{s}_n \times \mathfrak{u}_n$ defined by: $(A, A') \in \mathcal{X}$ if and only if A and A' have the same invariants. Since as we discussed above, the map that maps A to its collection of invariants is a definable map, this is a definable subassignment (note that it has a natural projection to \mathcal{P}). We have shown that the difference of the left-hand side and the right-hand side of Equation (1) in Conjecture 1.1.1 is a constructible motivic function on \mathcal{X} . Therefore, the Transfer Principle applies to it.

By inspection, all the ingredients of all the other variants of Conjecture 1.1.1 and Conjecture 1.1.2 are definable in the language $\mathcal{L}_{\mathcal{O}}$, and hence the Transfer Principle applies in all these cases.

REFERENCES

- [1] Raf Cluckers and François Loeser, *Ax-Kochen-Eršov theorems for p -adic integrals and motivic integration*, Geometric methods in algebra and number theory, Progr. Math., vol. 235, Birkhäuser Boston, Boston, MA, 2005, pp. 109–137.
- [2] Raf Cluckers and François Loeser, *Constructible motivic functions and motivic integration*, Invent. Math. **173** (2008), no. 1, 23–121.
- [3] ———, *Constructible exponential functions, motivic Fourier transform and transfer principle*, Annals of Mathematics **171** (2010), no. 2, 1011–1065.
- [4] R. Cluckers, T. Hales, and F. Loeser, *Transfer principle for the Fundamental Lemma*, available at [arXiv:0712.0708](https://arxiv.org/abs/0712.0708). To appear.
- [5] Jan Denef and François Loeser, *Definable sets, motives and p -adic integrals*, J. Amer. Math. Soc **14** (2001), no. 2, 429–469.

- [6] Hervé Jacquet and Stephen Rallis, *On the Gross-Prasad conjecture for the unitary group in 3 variables*.
- [7] J. Oesterlé, *Réduction modulo p^n des sous-ensembles analytiques fermés de \mathbb{Z}_p^N* , Invent. Math. **66** (1982), no. 2, 325–341.
- [8] Zhiwei Yun, *Fundamental Lemma of Jacquet-Rallis in positive characteristics*, available at [arXiv:0901.0900](https://arxiv.org/abs/0901.0900).